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SINGLE EDGE CRACKS IN RECTANGULAR TENSILE SHEET

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ABSTRACT

The stress intensity factors for single edge cracks in rectangular tensile sheet are studied by using a complex variable form of analysis. The results for deep cracks show a considerable increase in the intensity factors as compared to the case of symmetric edge cracks. Accurate numerical values of the stress intensity factors for the appropriate range of parameters are included.

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INTRODUCTION

There has been considerable interest recently in obtaining accurate numerical values for the stress intensity factors in the vicinity of crack tips in tensile sheet. Previous to several recent solutions, investigators in the field of fracture mechanics relied considerably on Irwin's[1] approximation derived from Westergaard's[2] solution for a series of equally spaced colinear cracks in an infinite sheet.

Recently, one of the authors has succeeded [3,4] in obtaining accurate results for the case of symmetric edge cracks in rectangular tensile sheet by using a complex variable approach with a complex mapping function to describe the geometry. The flexibility of the approach was illustrated by the relative simplicity in which the length/width ratio parameter for finite rectangular sheet was included in the analysis. The success of the analysis relied heavily on an effective plan invented for the truncation of the series representation of the exact mapping function.

In this report, the technique is applied to the problem of a single edge crack in rectangular tensile sheet. The analysis is complicated somewhat by a certain loss of mathematical symmetry as compared with the previous work. For deep cracks the numerical results are considerably different from the previous cases thereby indicating the sensitivity of the stress intensity factor to bending.

INITIAL FORMULATION

The rectangular sheet under tension weakened by a single edge crack (Figure 1) will be considered as lying in the complex Z-plane, $Z = x + iy$, with the center of the sheet described by $Z = Z_o$ where Z_o is real.

The complex variable methods of Muskhelishvili[5] depend on the representation of the well-known Airy stress function $U(x,y)$ in terms of two analytic functions of the complex variable Z namely, $\varphi(Z)$ and $\Psi(Z)$, where

$$U(x,y) = \mathrm{Re}[Z\varphi(Z)+\int^{Z}\Psi(Z)dZ]. \qquad (1)$$

To facilitate the consideration of boundary conditions, an auxiliary complex plane, the ζ-plane, is introduced and a functional relationship

$$Z = \omega(\zeta) \tag{2}$$

is found such that the unit circle, $\zeta = \sigma = e^{i\theta}$, and its interior in the ζ-plane map into the boundary and interior, respectively, of the region in Figure 1. The mapping function $\omega(\zeta)$ is analytic interior to the unit circle but contains singularities necessary to the description of corner points on the unit circle itself.

The stress functions $\varphi(Z)$ and $\Psi(Z)$ can be considered as functions of ζ. New notation can be minimized by designating $\varphi(Z) \equiv \varphi[\omega(\zeta)]$ as $\varphi(\zeta)$, etc., which leads to such definitions as $\varphi'(Z) = \varphi'(\zeta)/\omega'(\zeta)$, etc. (Primes are used to denote differentiation.) Thus, the stresses and displacements in rectangular coordinates can be written as

$$\sigma_y + \sigma_x = 4 \operatorname{Re}[\varphi'(\zeta)/\omega'(\zeta)] , \tag{3}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2 \{\overline{\omega(\zeta)}\, [\varphi'(\zeta)/\omega'(\zeta)]' + \Psi'(\zeta)\}/\omega'(\zeta) \tag{4}$$

$$2\mu\, (u+iv) = \eta\varphi(\zeta) - \omega(\zeta)\, \overline{\varphi'(\zeta)}/\overline{\omega'(\zeta)} - \overline{\Psi(\zeta)} \tag{5}$$

where μ, η are constants depending on the material and bars denote complex conjugates.

It is the primary concern of this paper to determine the stress intensity factor at the crack tip. The stress intensity factor K is defined by

$$\sigma_{max} = \sigma_y \approx \frac{K}{\sqrt{2r}}\, (\cos t/2)\, [1 + (\sin t/2)(\sin 3t/2)] \tag{6}$$

where (r,t) are polar coordinates defined with the origin at the crack tip It can be shown[6] that in terms of the present formulation,

$$K = 2\,\varphi'(1)/\sqrt{\omega'(1)}\ . \tag{7}$$

Thus, the major effort will involve determining the stress function $\varphi(\zeta)$ and finding the numerical values of $\varphi'(1)$.

THE MAPPING FUNCTION

By application of the Schwartz-Christoffel transformation, it was found that the required mapping function corresponds to an appropriate branch of

$$Z = \omega(\zeta) = \int_0^\zeta \frac{(\zeta-1)\,d\zeta}{\sqrt{(\zeta^2-2\zeta\cos\beta+1)(\zeta^2-2\zeta\cos\alpha+1)(\zeta^2+1)}}\ . \tag{8}$$

In Equation 8, $0 < \beta < \alpha < \pi/2$, where α and β can be varied to obtain desired ratios y_o/x_o and L/x_o. The choice of branch was made by defining $\omega(1) = Z_o - x_o + L$, thus

$$\omega(e^{i\alpha}) = Z_o - x_o - iy_o, \qquad \omega(e^{i\beta}) = Z_o - x_o$$

$$\omega(e^{i\pi/2}) = Z_o + x_o - iy_o, \text{ etc.}$$

By considering the mapping function on the unit circle, the length parameters can be expressed in terms of standard elliptic integrals of the first kind, $F(\varphi,K)$ [Reference 7], as follows:

$$x_o = \frac{1}{2}\,[\cos\beta\,(1+\cos\alpha)]^{-1/2}\,F(\pi/2,\,k_1), \tag{9}$$

$$y_o = [\cos\beta(1+\cos\alpha)]^{-1/2}\,F(\pi/2,\,k_2), \tag{10}$$

$$L = [\cos\beta\,(1+\cos\alpha)]^{-1/2}\,F(\varphi_1,\,k_1)\ , \tag{11}$$

where

$$\varphi_1 = \sin^{-1} [(1+\cos\alpha)(1-\cos\beta)/(1-\cos\alpha)(1+\cos\beta)]^{1/2},$$

$$k_1 = [\cos\alpha\ (1+\cos\beta)/\cos\beta\ (1+\cos\alpha)]^{1/2},$$

$$k_s = [(\cos\beta-\cos\alpha)/\cos\beta(1+\cos\alpha)]^{1/2}.$$

It is convenient to define φ_s by the relation

$$\varphi_s = \sin^{-1} k_1 .$$

Then it follows immediately that

$$k_s = \cos\varphi_s .$$

The length/width ratio of the plate now can be written as

$$y_o/x_o = 2\,\frac{F(\sin\varphi_s,\ \pi/2)}{F(\cos\varphi_s,\ \pi/2)} . \qquad (12)$$

It is clear from (12) that a fixed y_o/x_o ratio determines φ_s and thus a fixed relation between α and β. This computation can be carried out by simply interpolating from existing tables.[7] Variation of β, say, then permits a study of various L/x_o ratios.

The mapping function has six branch points falling on the unit circle (Figure 2). In addition to these corner-describing singularities, the crack tip is described by the root of $\omega'(\sigma) = 0$ which occurs at $\sigma = 1$.

The mapping function can be expressed in series form as

$$Z = \omega(\zeta) = \sum_{n=1}^{\infty} A_n\, \zeta^n \qquad (13)$$

where the mapping coefficients are real. Although the A_n's can be related to integrals involving the elliptic functions, it is simpler computationally to calculate them from easily derived recursive formulae.

It was found that retention of the exact mapping function and power series representation of the stress functions leads to the necessity of solving an infinite system of linear equations. The truncation procedure developed[3] was used in order to achieve an effective rate of convergence.

In general, the procedure involves the use of selected truncations of the series (13) preserving the properties of the first and second derivatives, $\omega'(\zeta)$ and $\omega''(\zeta)$, at the crack tip, namely at $\zeta = 1$.

In particular, we define

$$\omega_T(\zeta) = \sum_{n=1}^{m+2} \epsilon_n \zeta^n \tag{14}$$

where

$$\epsilon_n = A_n, \; n = 1, 2, \; \ldots\ldots\; m; \tag{15}$$

$$\epsilon_{m+1} = R, \quad \epsilon_{m+2} = S$$

and

$$(m+2)S = Q - S_m + mT_m$$
$$(m+1)R = -Q + S_m - (m+1)\, T_m \tag{16}$$

where

$$T_m = \sum_{n=1}^{m} n\, \epsilon_n\,, \qquad S_m = \sum_{n=1}^{m} n(n-1)\, \epsilon_n$$

$$Q = \omega''(1) = (1/2)\left(\frac{1}{\sqrt{2(1-\cos\beta)(1-\cos\alpha)}} \right).$$

With this definition of $\omega_T(\zeta)$, satisfaction of $\omega_T'(1) = 0$ and $\omega''_T(1) = Q$ is assured. The choice of m is made from an examination of the partial sums

derived from the exact mapping function and selecting those values of m for which $\omega'(1) \approx 0$ and $\omega''(1) \approx 0$ simultaneously. Thus, the necessary corrections of the geometry in the vicinity of the crack tip can be made with little disturbance of the over-all configuration.

DETERMINATION OF THE STRESS FUNCTIONS

The loading conditions in Figure 1 can be expressed conveniently in terms of the force resultant. Considering an arc of the material with element dS, we denote the horizontal and vertical forces as XdS and YdS, respectively. If the arc is taken as the boundary of Figure 1, then S can be considered as a function of σ. The boundary condition can be written as

$$\varphi(\sigma) + \omega(\sigma)\,\overline{\varphi'(\sigma)}/\overline{\omega'(\sigma)} + \overline{\Psi(\sigma)} = \int^{S}(X+iY)dS = g(\sigma). \tag{17}$$

The solution then requires the determination of the functions $\varphi(\zeta)$ and $\Psi(\zeta)$ which are analytic for $|\zeta| < 1$ and satisfy the loading conditions (Equation 17).

The problem will be considered in terms of polynomial approximations of the exact geometry as indicated by the truncation procedure of the previous paragraph.

It was shown in Reference 3 that the stress functions $\varphi(\zeta)$, $\Psi(\zeta)$ have the following general structures:

$$\varphi(\zeta) = (T/4)\,\omega(\zeta) + \varphi_1(\zeta) \tag{18}$$

$$\Psi(\zeta) = (T/2)\,\omega(\zeta) + \Psi_1(\zeta) \tag{19}$$

where

$$\varphi_1(\zeta) = T\left[\sum_{n=1}^{\infty} C_n\zeta^n + \sum_{n=1}^{m+2} \alpha_n\zeta^n\right]$$

$$\Psi_1(\zeta) = T\left[A/(\zeta-1) + \sum_{n=0}^{\infty} \beta_n\zeta^n\right]$$

and

$$A = \frac{-\omega_T(1)\,\varphi_1{}'(1)}{\omega''_T(1)}.$$

With the above forms of $\varphi(\zeta)$, $\Psi(\zeta)$, the boundary condition (17) may be written as follows:

$$\varphi_1(\sigma) + \omega(\sigma)\,\overline{\varphi_1'(\sigma)}\,/\,\overline{\omega'(\sigma)} + \overline{\Psi_1(\sigma)} = g(\sigma) - T/2[\omega(\sigma)+\overline{\omega(\sigma)}] = G(\sigma). \tag{20}$$

If $G(\sigma)$ is expanded in the Fourier series,

$$G(\sigma) = \sum_{K=-\infty}^{\infty} C_K\,\sigma^K, \tag{21}$$

then,

$$C_K = \frac{T}{2\pi}\left[\{\omega(e^{i\beta})+\overline{\omega(e^{i\beta})}\}\,\frac{\sin K\beta}{K} - \int_{-\beta}^{\beta}\left(\sum_{n=1}^{n+2} A_n \cos n\theta\right)(\cos K\theta - i \sin K\theta)\,d\theta\right]$$

$$C_K = C_{-K} \quad \text{for } K = 1, 2, \ldots\ldots\ldots \infty .$$

Insertion of the appropriate series expansions into Equation 20 and equating coefficients of equal powers of σ (from the positive powers), one finds the system for determination of α_n to be

$$\sum_{n=1}^{m+3-p} n(\epsilon_n \alpha_{n+p-1} + C_n\,\epsilon_{n+p-1} + \alpha_n\,\epsilon_{n+p-1}) = 0,$$

$$p = 1, 2, \ldots\ldots\ldots, m+2. \tag{22}$$

The calculation of the stress intensity factor K requires the determination of $\varphi'(1)$ as indicated in Equation 7

From Equation 18 it can be seen that

$$\varphi'(1) = T\left[\sum_{n=1}^{\infty} n\,C_n + \sum_{n=1}^{m+2} n\,\alpha_n\right]$$

$$= T\,[\Sigma_1 + \Sigma_2]\ . \tag{23}$$

Σ_1 can be determined directly from the Fourier coefficients of the applied load. Σ_2 can be calculated from the solution of the system of Equations 22.

The analysis was carried out numerically for the two cases $y_o/x_o = 1.0$ and $y_o/x_o = 3.06$. In general, three values of m were selected to study the convergence of Σ_1 and Σ_2. The range of m considered fell in the intervals $0 < m_1 < 50$, $50 < m_2 < 100$, $100 < m_3 < 150$. A study of the convergence of Σ_1 and Σ_2 led to the assertion that the calculated values of $\varphi'(1)/T$ are numerically correct to within 2 percent.

The key data are presented in Table I below:

TABLE I

EVALUATION OF K/T

α	β	x_o	y_o/x_o	$L/2x_o$	$\varphi'(1)T$	K/T	K/K_A
0.3706	0.1600	1.150	0.997	0.144	1.19	0.69	1.19
0.4445	0.3000	1.178	0.997	0.254	1.33	1.15	1.45
0.7573	0.7000	1.381	0.997	0.487	1.70	2.88	2.22
1.0279	1.0000	1.754	0.997	0.621	2.49	5.75	3.17
1.2151	1.2000	2.271	0.997	0.711	3.74	10.11	4.16
...	...	...	...	...	...	...	...
1.5095	0.5000	0.841	3.056	0.170	0.331	0.64	1.18
1.5255	1.0000	1.080	3.056	0.378	0.604	1.65	1.71
1.5368	1.2000	1.326	3.056	0.490	0.954	2.84	2.22
1.5525	1.4000	1.951	3.056	0.647	2.154	6.88	3.44

The final column K/K_A is presented for the sake of comparison with Irwin's approximation for symmetric edge cracks. In the present notation,

$$K_A/T = \left[\frac{4x_o}{\pi} \tan \frac{\pi L}{4x_o}\right]^{1/2} . \qquad (24)$$

The ratio K/K_A as a function of $L/2x_o$ is shown in Figure 3. Its departure from unity is dominantly due to the bending effect of the applied load.

DISCUSSION

For very short crack lengths one would expect the solution to behave as if the plate were semi-infinite in extent. This is consistent with the numerical result $K/K_A \to 1.13$ as $L \to 0$ in Figure 3.

Comparison of the results for $y_o/x_o = 1.00$ and $y_o/x_o = 3.06$ indicates agreement for very short and fairly long crack depths. The variation in the interval $0.2 < L/2x_o < 0.5$ is due to the difference in length/width ratios. It is reasonable to expect that the results would not change for larger y_o/x_o ratios.

ACKNOWLEDGMENT

The authors are indebted to Mr. J. Adachi for illuminating discussions relative to estimating the bending effect.

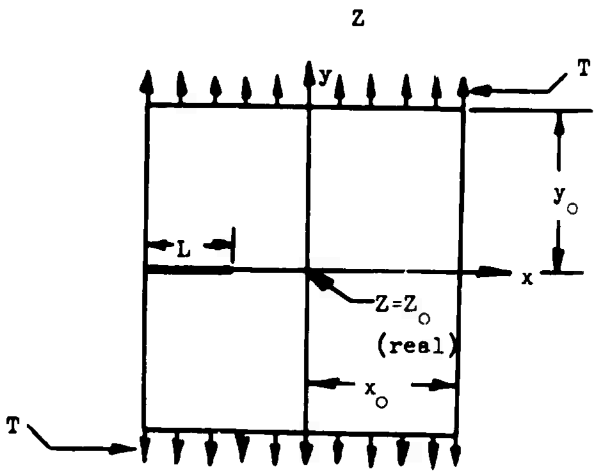

Figure 1. REGION IN Z-PLANE

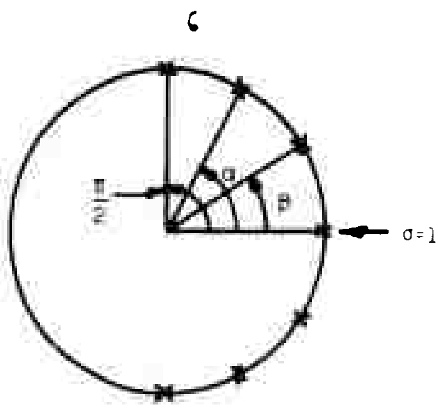

Figure 2. THE UNIT CIRCLE IN THE ζ-PLANE

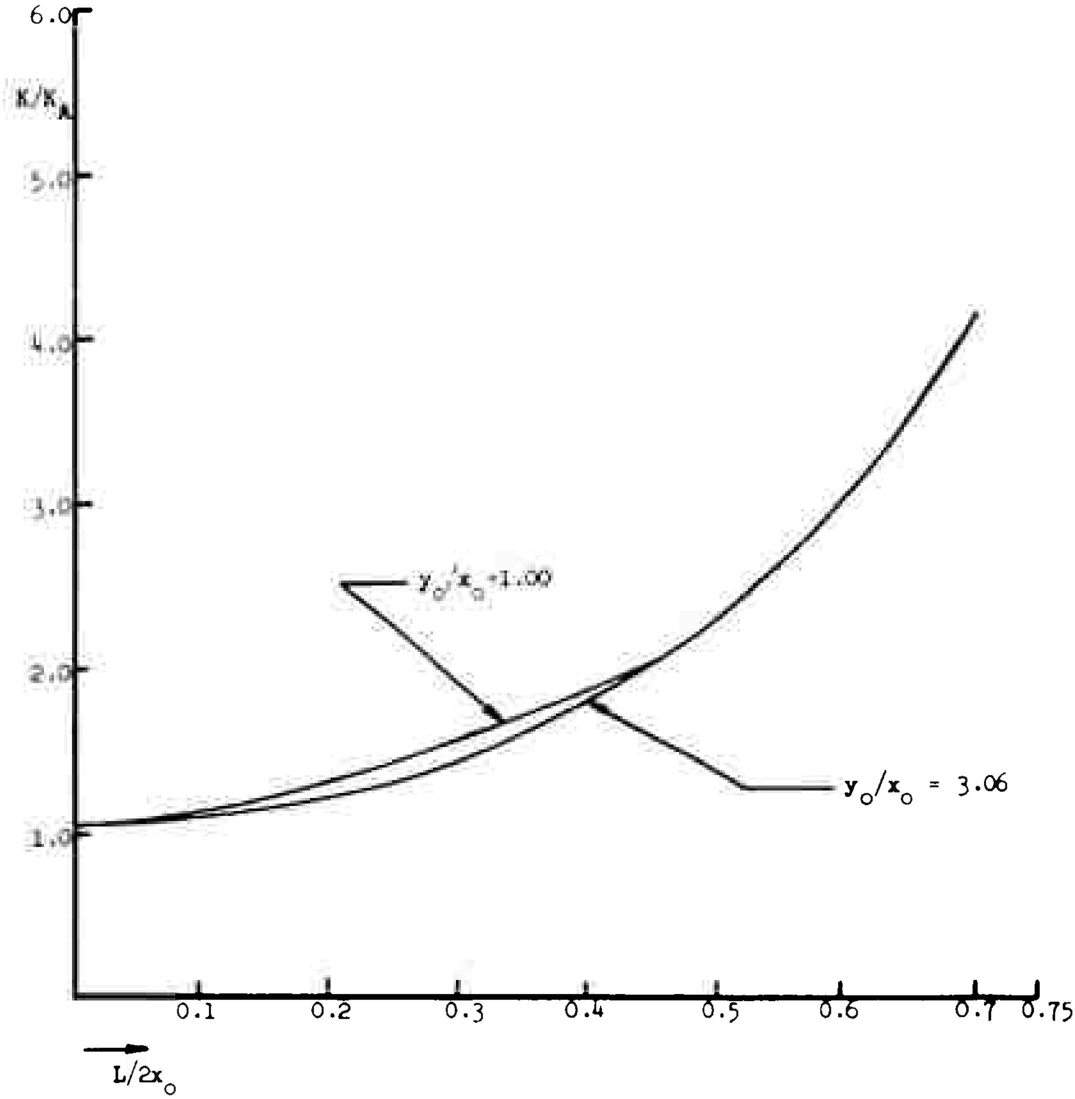

Figure 3. VARIATION OF STRESS INTENSITY FACTORS WITH CRACK LENGTH

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